

# LOCALIZATIONS OF INDUCTIVELY FACTORED ARRANGEMENTS

TILMAN MÖLLER AND GERHARD RÖHRLE

ABSTRACT. We show that the class of inductively factored arrangements is closed under taking localizations. We illustrate the usefulness of this with an application.

## 1. INTRODUCTION

The notion of a nice arrangement is due to Terao [Ter92]. This class generalizes the class of supersolvable arrangements, [OST84] (cf. [OT92, Thm. 3.81]). There is an inductive version of this class, so called inductively factored arrangements, due to Jambu and Paris [JP95], see Definition 2.7. This inductive class (properly) contains the class of supersolvable arrangements and is (properly) contained in the class of inductively free arrangements, see [HR16a, Rem. 3.33].

For an overview on properties of nice and inductively factored arrangements, and for their connection with the underlying Orlik-Solomon algebra, see [OT92, §3], [JP95], and [HR16a]. In [HR16a], Hoge and the second author proved an addition-deletion theorem for nice arrangements, see Theorem 2.6 below. This is an analogue of Terao's celebrated Addition-Deletion Theorem 2.1 for free arrangements for the class of nice arrangements.

The class of free arrangements is known to be closed under taking localizations, [OT92, Thm. 4.37]. It is also known that this property restricts to various stronger notions of freeness, see [HRS16, Thm. 1.1]. It is clear that the class of nice arrangements also satisfies this property, see Remark 2.5(ii) below. Therefore, it is natural to investigate this question for the stronger property of inductively factored arrangements as well. Here is the main result of our note.

**Theorem 1.1.** *The class of inductively factored arrangements is closed under taking localizations.*

Theorem 1.1 readily extends to the class of hereditarily inductively factored arrangements, see Remark 3.7. Also, we give a short example to show the utility of such a result.

## 2. RECOLLECTIONS AND PRELIMINARIES

**2.1. Hyperplane arrangements.** Let  $\mathbb{K}$  be a field and let  $V = \mathbb{K}^\ell$  be an  $\ell$ -dimensional  $\mathbb{K}$ -vector space. A *hyperplane arrangement*  $\mathcal{A}$  in  $V$  is a finite collection of hyperplanes in  $V$ . We also use the term  $\ell$ -arrangement for  $\mathcal{A}$ .

---

2010 *Mathematics Subject Classification.* Primary 52C35, 14N20; Secondary 51D20.

*Key words and phrases.* nice arrangement, inductively factored arrangement, localization of an arrangement.

The *lattice*  $L(\mathcal{A})$  of  $\mathcal{A}$  is the set of subspaces of  $V$  of the form  $H_1 \cap \dots \cap H_i$  where  $\{H_1, \dots, H_i\}$  is a subset of  $\mathcal{A}$ . For  $X \in L(\mathcal{A})$ , we have two associated arrangements, firstly  $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$ , the *localization of  $\mathcal{A}$  at  $X$* , and secondly, the *restriction of  $\mathcal{A}$  to  $X$* ,  $(\mathcal{A}^X, X)$ , where  $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$ . Note that  $V$  belongs to  $L(\mathcal{A})$  as the intersection of the empty collection of hyperplanes and  $\mathcal{A}^V = \mathcal{A}$ . The lattice  $L(\mathcal{A})$  is a partially ordered set by reverse inclusion:  $X \leq Y$  provided  $Y \subseteq X$  for  $X, Y \in L(\mathcal{A})$ .

If  $0 \in H$  for each  $H$  in  $\mathcal{A}$ , then  $\mathcal{A}$  is called *central*. If  $\mathcal{A}$  is central, then the *center*  $T_{\mathcal{A}} := \bigcap_{H \in \mathcal{A}} H$  of  $\mathcal{A}$  is the unique maximal element in  $L(\mathcal{A})$  with respect to the partial order. We have a *rank* function on  $L(\mathcal{A})$ :  $r(X) := \text{codim}_V(X)$ . The *rank*  $r := r(\mathcal{A})$  of  $\mathcal{A}$  is the rank of a maximal element in  $L(\mathcal{A})$ . Throughout, we only consider central arrangements.

More generally, for  $U$  an arbitrary subspace of  $V$ , we can define  $\mathcal{A}_U := \{H \in \mathcal{A} \mid U \subseteq H\} \subseteq \mathcal{A}$ , the *localization of  $\mathcal{A}$  at  $U$* , and  $\mathcal{A}^U := \{U \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_U\}$ , a subarrangement in  $U$ .

**2.2. Free hyperplane arrangements.** Let  $S = S(V^*)$  be the symmetric algebra of the dual space  $V^*$  of  $V$ . Let  $\text{Der}(S)$  be the  $S$ -module of  $\mathbb{K}$ -derivations of  $S$ . Since  $S$  is graded,  $\text{Der}(S)$  is a graded  $S$ -module.

Let  $\mathcal{A}$  be an arrangement in  $V$ . Then for  $H \in \mathcal{A}$  we fix  $\alpha_H \in V^*$  with  $H = \ker \alpha_H$ . The *defining polynomial*  $Q(\mathcal{A})$  of  $\mathcal{A}$  is given by  $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$ . The *module of  $\mathcal{A}$ -derivations* of  $\mathcal{A}$  is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

We say that  $\mathcal{A}$  is *free* if  $D(\mathcal{A})$  is a free  $S$ -module, cf. [OT92, §4].

If  $\mathcal{A}$  is a free arrangement, then the  $S$ -module  $D(\mathcal{A})$  admits a basis of  $n$  homogeneous derivations, say  $\theta_1, \dots, \theta_n$ , [OT92, Prop. 4.18]. While the  $\theta_i$ 's are not unique, their polynomial degrees  $\text{pdeg } \theta_i$  are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement  $\mathcal{A}$  and is denoted by  $\exp \mathcal{A}$ .

Terao's celebrated *Addition-Deletion Theorem* which we recall next plays a pivotal role in the study of free arrangements, [OT92, §4]. For  $\mathcal{A}$  non-empty, let  $H_0 \in \mathcal{A}$ . Define  $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ , and  $\mathcal{A}'' := \mathcal{A}^{H_0} = \{H_0 \cap H \mid H \in \mathcal{A}'\}$ , the restriction of  $\mathcal{A}$  to  $H_0$ . Then  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  is a *triple* of arrangements, [OT92, Def. 1.14].

**Theorem 2.1** ([Ter80]). *Suppose that  $\mathcal{A}$  is a non-empty  $\ell$ -arrangement. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple of arrangements. Then any two of the following statements imply the third:*

- (i)  $\mathcal{A}$  is free with  $\exp \mathcal{A} = \{b_1, \dots, b_{\ell-1}, b_{\ell}\}$ ;
- (ii)  $\mathcal{A}'$  is free with  $\exp \mathcal{A}' = \{b_1, \dots, b_{\ell-1}, b_{\ell} - 1\}$ ;
- (iii)  $\mathcal{A}''$  is free with  $\exp \mathcal{A}'' = \{b_1, \dots, b_{\ell-1}\}$ .

There are various stronger notions of freeness which we discuss in the following subsections.

**2.3. Inductively free arrangements.** Theorem 2.1 motivates the notion of *inductively free* arrangements, see [Ter80] or [OT92, Def. 4.53].

**Definition 2.2.** The class  $\mathcal{IF}$  of *inductively free* arrangements is the smallest class of arrangements subject to

- (i)  $\Phi_\ell \in \mathcal{IF}$  for each  $\ell \geq 0$ ;
- (ii) if there exists a hyperplane  $H_0 \in \mathcal{A}$  such that both  $\mathcal{A}'$  and  $\mathcal{A}''$  belong to  $\mathcal{IF}$ , and  $\exp \mathcal{A}'' \subseteq \exp \mathcal{A}'$ , then  $\mathcal{A}$  also belongs to  $\mathcal{IF}$ .

Free arrangements are closed with respect to taking localizations, cf. [OT92, Thm. 4.37]. This also holds for the class  $\mathcal{IF}$ .

**Theorem 2.3** ([HRS16, Thm. 1.1]). *If  $\mathcal{A}$  is inductively free, then so is  $\mathcal{A}_U$  for every subspace  $U$  in  $V$ .*

**2.4. Nice and inductively factored arrangements.** The notion of a *nice* or *factored* arrangement goes back to Terao [Ter92]. It generalizes the concept of a supersolvable arrangement, see [OST84, Thm. 5.3] and [OT92, Prop. 2.67, Thm. 3.81]. Terao's main motivation was to give a general combinatorial framework to deduce factorizations of the underlying Orlik-Solomon algebra, see also [OT92, §3.3]. We recall the relevant notions from [Ter92] (cf. [OT92, §2.3]):

**Definition 2.4.** Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ .

- (a)  $\pi$  is called *independent*, provided for any choice  $H_i \in \pi_i$  for  $1 \leq i \leq s$ , the resulting  $s$  hyperplanes are linearly independent, i.e.  $r(H_1 \cap \dots \cap H_s) = s$ .
- (b) Let  $X \in L(\mathcal{A})$ . The *induced partition*  $\pi_X$  of  $\mathcal{A}_X$  is given by the non-empty blocks of the form  $\pi_i \cap \mathcal{A}_X$ .
- (c)  $\pi$  is *nice* for  $\mathcal{A}$  or a *factorization* of  $\mathcal{A}$  provided
  - (i)  $\pi$  is independent, and
  - (ii) for each  $X \in L(\mathcal{A}) \setminus \{V\}$ , the induced partition  $\pi_X$  admits a block which is a singleton.

If  $\mathcal{A}$  admits a factorization, then we also say that  $\mathcal{A}$  is *factored* or *nice*.

**Remark 2.5.** The class of nice arrangements is closed under taking localizations. For, if  $\mathcal{A}$  is non-empty and  $\pi$  is a nice partition of  $\mathcal{A}$ , then the non-empty parts of the induced partition  $\pi_X$  form a nice partition of  $\mathcal{A}_X$  for each  $X \in L(\mathcal{A}) \setminus \{V\}$ ; cf. the proof of [Ter92, Cor. 2.11].

Following Jambu and Paris [JP95], we introduce further notation. Suppose  $\mathcal{A}$  is not empty. Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ . Let  $H_0 \in \pi_1$  and let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be the triple associated with  $H_0$ . Then  $\pi$  induces a partition  $\pi'$  of  $\mathcal{A}'$ , i.e. the non-empty subsets  $\pi_i \cap \mathcal{A}'$ . Note that since  $H_0 \in \pi_1$ , we have  $\pi_i \cap \mathcal{A}' = \pi_i$  for  $i = 2, \dots, s$ . Also, associated with  $\pi$  and  $H_0$ , we define the *restriction map*

$$\varrho := \varrho_{\pi, H_0} : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}'' \text{ given by } H \mapsto H \cap H_0$$

and set

$$\pi_i'' := \varrho(\pi_i) = \{H \cap H_0 \mid H \in \pi_i\} \text{ for } 2 \leq i \leq s.$$

In general,  $\varrho$  need not be surjective nor injective. However, since we are only concerned with cases when  $\pi'' = (\pi_2'', \dots, \pi_s'')$  is a partition of  $\mathcal{A}''$ ,  $\varrho$  has to be onto and  $\varrho(\pi_i) \cap \varrho(\pi_j) = \emptyset$  for  $i \neq j$ .

The following analogue of Terao's Addition-Deletion Theorem 2.1 for free arrangements for the class of nice arrangements is proved in [HR16a, Thm. 3.5].

**Theorem 2.6.** *Suppose that  $\mathcal{A} \neq \Phi_\ell$ . Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ . Let  $H_0 \in \pi_1$  and let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be the triple associated with  $H_0$ . Then any two of the following statements imply the third:*

- (i)  $\pi$  is nice for  $\mathcal{A}$ ;
- (ii)  $\pi'$  is nice for  $\mathcal{A}'$ ;
- (iii)  $\varrho : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$  is bijective and  $\pi''$  is nice for  $\mathcal{A}''$ .

Note the bijectivity condition on  $\varrho$  in Theorem 2.6 is necessary, cf. [HR16a, Ex. 3.3]. Theorem 2.6 motivates the following stronger notion of factorization, cf. [JP95], [HR16a, Def. 3.8].

**Definition 2.7.** The class  $\mathcal{IFAC}$  of *inductively factored* arrangements is the smallest class of pairs  $(\mathcal{A}, \pi)$  of arrangements  $\mathcal{A}$  together with a partition  $\pi$  subject to

- (i)  $(\Phi_\ell, (\emptyset)) \in \mathcal{IFAC}$  for each  $\ell \geq 0$ ;
- (ii) if there exists a partition  $\pi$  of  $\mathcal{A}$  and a hyperplane  $H_0 \in \pi_1$  such that for the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  associated with  $H_0$  the restriction map  $\varrho = \varrho_{\pi, H_0} : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$  is bijective and for the induced partitions  $\pi'$  of  $\mathcal{A}'$  and  $\pi''$  of  $\mathcal{A}''$  both  $(\mathcal{A}', \pi')$  and  $(\mathcal{A}'', \pi'')$  belong to  $\mathcal{IFAC}$ , then  $(\mathcal{A}, \pi)$  also belongs to  $\mathcal{IFAC}$ .

If  $(\mathcal{A}, \pi)$  is in  $\mathcal{IFAC}$ , then we say that  $\mathcal{A}$  is *inductively factored with respect to  $\pi$* , or else that  $\pi$  is an *inductive factorization* of  $\mathcal{A}$ . Sometimes we simply say  $\mathcal{A}$  is *inductively factored* without reference to a specific inductive factorization of  $\mathcal{A}$ .

**Remark 2.8.** If  $\pi$  is an inductive factorization of  $\mathcal{A}$ , then there exists an *induction of factorizations* by means of Theorem 2.6 as follows. This procedure amounts to choosing a total order on  $\mathcal{A}$ , say  $\mathcal{A} = \{H_1, \dots, H_n\}$ , so that each of the pairs  $(\mathcal{A}_0 = \Phi_\ell, \emptyset)$ ,  $(\mathcal{A}_i := \{H_1, \dots, H_i\}, \pi_i := \pi|_{\mathcal{A}_i})$ , and  $(\mathcal{A}_i'' := \mathcal{A}_i^{H_i}, \pi_i'')$  for each  $1 \leq i \leq n$ , belongs to  $\mathcal{IFAC}$  see [HR16a, Rem. 3.16].

The connection with the previous notions is as follows.

**Proposition 2.9** ([HR16a, Prop. 3.11]). *If  $\mathcal{A}$  is supersolvable, then  $\mathcal{A}$  is inductively factored.*

**Proposition 2.10** ([JP95, Prop. 2.2], [HR16a, Prop. 3.14]). *Let  $\pi = (\pi_1, \dots, \pi_r)$  be an inductive factorization of  $\mathcal{A}$ . Then  $\mathcal{A}$  is inductively free with  $\exp \mathcal{A} = \{0^{\ell-r}, |\pi_1|, \dots, |\pi_r|\}$ .*

**Definition 2.11.** We say that  $\mathcal{A}$  is *hereditarily inductively factored* provided  $\mathcal{A}^Y$  is inductively factored for every  $Y \in L(\mathcal{A})$ .

### 3. PROOF OF THEOREM 1.1

We readily reduce to the case where we localize with respect to a space  $X$  belonging to the intersection lattice of  $\mathcal{A}$ . For, letting  $X = \cap_{H \in \mathcal{A}_U} H \in L(\mathcal{A})$ , we have  $\mathcal{A}_X = \mathcal{A}_U$ .

We are going to show that if  $\pi$  is an inductive factorization of  $\mathcal{A}$ , then the restriction  $\pi_X$  of  $\pi$  to  $\mathcal{A}_X$  is an inductive factorization of the latter. We argue by induction on the rank  $r(\mathcal{A})$ . If  $r(\mathcal{A}) \leq 3$ , then  $r(\mathcal{A}_X) \leq 2$  for  $X \neq T_{\mathcal{A}}$ , so the result follows from the proof of Proposition 2.9 (and the fact that  $V < H < X = T(\mathcal{A}_X)$  is a maximal chain of modular elements in  $L(\mathcal{A}_X)$  for every  $H \in L(\mathcal{A}_X)$ ).

So suppose  $\mathcal{A}$  is inductively factored of rank  $r > 3$  and that the statement above holds for all inductively factored arrangements of rank less than  $r$ . Let  $\pi$  be an inductive factorization of  $\mathcal{A}$ . Let  $\{H_1, \dots, H_n\}$  be the total order on  $\mathcal{A}$  such that for  $\mathcal{A}_i := \{H_1, \dots, H_i\}$  the induced partition  $\pi_i := \pi|_{\mathcal{A}_i}$  is an inductive factorization of  $\mathcal{A}_i$  for  $i = 1, \dots, n$ , see Remark 2.8. Consider the sequence of inductive factorizations

$$(3.1) \quad (\mathcal{A}_1, \pi_1), (\mathcal{A}_2, \pi_2), \dots, (\mathcal{A}_n, \pi_n) = (\mathcal{A}, \pi).$$

Then, for  $i = 1, \dots, n$ , we have

$$(3.2) \quad \mathcal{A}_X \cap \mathcal{A}_i = (\mathcal{A}_i)_X.$$

For  $H \in \mathcal{A}_X \cap \mathcal{A}_i$ , we have  $H \leq X$ , and so by (3.2), for  $i = 1, \dots, n$ ,

$$(3.3) \quad (\mathcal{A}_X \cap \mathcal{A}_i)^H = ((\mathcal{A}_i)_X)^H = (\mathcal{A}_i^H)_X.$$

Consequently, localizing each member  $(\mathcal{A}_i, \pi_i)$  of the sequence (3.1) at  $X$ , removing redundant terms if necessary and reindexing the resulting distinct arrangements, we obtain the following sequence of subarrangements of  $\mathcal{A}_X$ ,

$$(3.4) \quad \mathcal{A}_{1,X} \subsetneq \mathcal{A}_{2,X} \subsetneq \dots \subsetneq \mathcal{A}_{m,X} = \mathcal{A}_X,$$

where  $\mathcal{A}_{i,X}$  is short for  $(\mathcal{A}_i)_X$ . In particular,  $|\mathcal{A}_{i,X}| = i$  and  $m \leq n$ . Thus we obtain the following sequence of subarrangements of  $\mathcal{A}_X$  along with induced partitions:

$$(3.5) \quad (\mathcal{A}_{1,X}, \pi_{1,X}), (\mathcal{A}_{2,X}, \pi_{2,X}), \dots, (\mathcal{A}_{m,X}, \pi_{m,X}) = (\mathcal{A}_X, \pi_X),$$

where  $\pi_{i,X}$  is the induced partition of  $\pi_i$  on  $\mathcal{A}_{i,X}$ , i.e.  $\pi_{i,X} := \pi|_{\mathcal{A}_{i,X}}$ . We claim that (3.5) is an inductive chain of factorizations of  $\mathcal{A}_X$ , so that  $\pi_X$  is an inductive factorization of  $\mathcal{A}_X$ .

Now let  $H_i \in \mathcal{A}_X \cap \mathcal{A}_i = \mathcal{A}_{i,X}$  be the relevant hyperplane in the  $i$ th step in the sequence (3.4). Let  $(\mathcal{A}_{i,X}, \mathcal{A}'_{i,X}, \mathcal{A}''_{i,X})$  be the triple with respect to  $H_i$ . Thus, by the constructions of the chains in (3.4) and (3.5), we have for each  $2 \leq i \leq m$

$$(3.6) \quad \mathcal{A}'_{i,X} = \mathcal{A}_{i-1,X} \quad \text{and} \quad \pi'_{i,X} = \pi_{i-1,X}.$$

Without loss, suppose  $H_i$  belongs to the first part  $(\pi_{i,X})_1$  of  $\pi_{i,X}$  for each  $i$ . Thanks to (3.6), Remark 2.5 and Theorem 2.6 the corresponding restriction map  $\varrho : \mathcal{A}_{i,X} \setminus (\pi_{i,X})_1 \rightarrow \mathcal{A}''_{i,X}$  is bijective for each  $i$ .

Since  $(\mathcal{A}''_i, \pi''_i)$  belongs to  $\mathcal{IFAC}$ , it follows by induction on the rank that each localization

$$(\mathcal{A}''_{i,X}, \pi''_{i,X}) := ((\mathcal{A}''_i)_X, (\pi''_i)_X)$$

also belongs to  $\mathcal{IFAC}$ , for each  $i < m$ , where we used (3.5) and set  $\pi''_{i,X} := (\pi''_i)_X$ .

Finally, since  $\mathcal{A}_{1,X}$  is of rank 1,  $(\mathcal{A}_{1,X}, \pi_{1,X})$  belongs to  $\mathcal{IFAC}$ . Thus, because  $(\mathcal{A}''_{i,X}, \pi''_{i,X})$  belongs to  $\mathcal{IFAC}$  and the corresponding restriction map  $\varrho : \mathcal{A}_{i,X} \setminus (\pi_{i,X})_1 \rightarrow \mathcal{A}''_{i,X}$  is bijective for each  $i < m$ , it follows from (3.6) and a repeated application of the addition part of Theorem 2.6 that also  $(\mathcal{A}_{m,X}, \pi_{m,X}) = (\mathcal{A}_X, \pi_X)$  belongs to  $\mathcal{IFAC}$ , as desired.

**Remark 3.7.** Theorem 1.1 readily extends to hereditarily inductively factored arrangements. For, let  $\mathcal{A}$  be hereditarily inductively factored and let  $Y \leq X$  in  $L(\mathcal{A})$ . Then, since  $\mathcal{A}^Y$  is inductively factored, so is  $(\mathcal{A}^Y)_X$ , by Theorem 1.1. Finally, since  $(\mathcal{A}_X)^Y = (\mathcal{A}^Y)_X$ , it follows that  $(\mathcal{A}_X)^Y$  is inductively factored.

The following example shows the utility of the results above.

**Example 3.8.** Let  $V = \mathbb{C}^\ell$  be an  $\ell$ -dimensional  $\mathbb{C}$ -vector space. Orlik and Solomon defined intermediate arrangements  $\mathcal{A}_\ell^k(r)$  in [OS82, §2] (cf. [OT92, §6.4]) which interpolate between the reflection arrangements  $\mathcal{A}(G(r, 1, \ell))$  and  $\mathcal{A}(G(r, r, \ell))$  of the complex reflection groups  $G(r, 1, \ell)$  and  $G(r, r, \ell)$ . For  $\ell, r \geq 2$  and  $0 \leq k \leq \ell$ , the defining polynomial of  $\mathcal{A}_\ell^k(r)$  is

$$Q(\mathcal{A}_\ell^k(r)) = x_1 \cdots x_k \prod_{\substack{1 \leq i < j \leq \ell \\ 0 \leq n < r}} (x_i - \zeta^n x_j),$$

where  $\zeta$  is a primitive  $r$ th root of unity, so that  $\mathcal{A}_\ell^\ell(r) = \mathcal{A}(G(r, 1, \ell))$  and  $\mathcal{A}_\ell^0(r) = \mathcal{A}(G(r, r, \ell))$ . Note that for  $1 < k < \ell$ ,  $\mathcal{A}_\ell^k(r)$  is not a reflection arrangement.

Each of these arrangements is known to be free, cf. [OT92, Prop. 6.85]. The supersolvable and inductively free cases among them were classified in [AHR14b], and [AHR14a], respectively.

If  $k \in \{\ell - 1, \ell\}$ , then  $\mathcal{A}_\ell^k(r)$  is supersolvable, by [AHR14b, Thm. 1.3], and so  $\mathcal{A}_\ell^k(r)$  is inductively factored, by Proposition 2.9. Let  $\ell \geq 4$ . We claim that  $\mathcal{A}_\ell^k(r)$  is not nice for  $0 \leq k \leq \ell - 4$  and moreover  $\mathcal{A}_\ell^k(r)$  is not inductively factored for  $0 \leq k \leq \ell - 3$ .

For  $k = 0$ , this follows from [HR16b, Thm. 1.3]. So let  $1 \leq k \leq \ell - 3$  and set  $\mathcal{A} = \mathcal{A}_\ell^k(r)$ . Define

$$X := \bigcap_{\substack{k+1 \leq i < j \leq \ell \\ 0 \leq n < r}} \ker(x_i - \zeta^n x_j).$$

Then one checks that

$$\mathcal{A}_X \cong \mathcal{A}_{\ell-k}^0(r) = \mathcal{A}(G(r, r, \ell - k)).$$

For  $1 \leq k \leq \ell - 4$ , it follows from [HR16b, Thm. 1.3] that  $\mathcal{A}(G(r, r, \ell - k))$  is not nice. Consequently, neither is  $\mathcal{A}_\ell^k(r)$ , by Remark 2.5. For  $k = \ell - 3$ , we have  $\mathcal{A}_X \cong \mathcal{A}(G(r, r, 3))$ . By [HR16b, Cor. 1.4], the latter is not inductively factored, thus neither is  $\mathcal{A}_\ell^{\ell-3}(r)$ , thanks to Theorem 1.1.

**Acknowledgments:** We acknowledge support from the DFG-priority program SPP1489 “Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory”.

## REFERENCES

- [AHR14a] N. Amend, T. Hoge and G. Röhrle, *On inductively free restrictions of reflection arrangements*, J. Algebra, **418** (2014), 197–212.
- [AHR14b] ———, *Supersolvable restrictions of reflection arrangements*, J. Combin. Theory Ser. A **127** (2014), 336–352.
- [HR16a] T. Hoge and G. Röhrle, *Addition-Deletion Theorems for Factorizations of Orlik-Solomon Algebras and nice Arrangements*, European J. Combin. **55** (2016), 20–40.
- [HR16b] ———, *Nice Reflection Arrangements*, <http://arxiv.org/abs/1505.04603>.
- [HRS16] T. Hoge, G. Röhrle, and A. Schauenburg, *Inductive and Recursive Freeness of Localizations of Multiarrangements*, <http://arxiv.org/abs/1501.06312>.
- [JP95] M. Jambu and L. Paris, *Combinatorics of Inductively Factored Arrangements*, European J. Combin. **16** (1995), 267–292.
- [OS82] P. Orlik and L. Solomon, *Arrangements Defined by Unitary Reflection Groups*, Math. Ann. **261**, (1982), 339–357.
- [OST84] P. Orlik, L. Solomon, and H. Terao, *Arrangements of hyperplanes and differential forms*. Combinatorics and algebra (Boulder, Colo., 1983), 29–65, Contemp. Math., **34**, Amer. Math. Soc., Providence, RI, 1984.
- [OT92] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.
- [Ter80] H. Terao, *Arrangements of hyperplanes and their freeness I, II*, J. Fac. Sci. Univ. Tokyo **27** (1980), 293–320.
- [Ter92] ———, *Factorizations of the Orlik-Solomon Algebras*, Adv. in Math. **92**, (1992), 45–53.

FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM, GERMANY

*E-mail address:* `tilman.moeller@rub.de`

*E-mail address:* `gerhard.roehrle@rub.de`